

Interplay of fixed points in scalar models

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We performed the renormalization group analysis of scalar models exhibiting spontaneous symmetry breaking. It is shown that an infrared fixed point appears inducing a dynamical scale, that can be identified with the correlation length of the model. This enables one to identify the type of the phase transition which shows similarity to the one appearing in the crossover scale.

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I. Introduction.— The renormalization group (RG) method [1] is one of the best candidate to take into account the degrees of freedom of a quantum field theoretical model systematically. The method enables us to identify the fixed points of the models and the scaling of the couplings in their vicinity which can give us the critical exponents of the corresponding fixed point [2–4]. The RG method is usually tested in 3-dimensional (3D) $O(N)$ scalar models, where a trivial Gaussian and a non-trivial Wilson-Fisher (WF) fixed point exist. The calculation of the critical exponents of the latter fixed point plays the test ground of all inventions in the RG method [2, 5–8]. The scaling of the correlation length ξ defines the exponent ν as

$$\xi \propto t^{-\nu}, \quad (1)$$

with the reduced temperature t . There is a second order, Ising type phase transition in the 3D $O(1)$ model with exponent $\nu = 0.63$. The Kosterlitz-Thouless (KT) or infinite order phase transition [9, 10] is widely investigated too, furthermore it gives a great challenge to recover the scaling of ξ according to

$$\log \xi \propto t^{-\nu}. \quad (2)$$

A typical example for KT transition is presented by the sine-Gordon (SG) model [4, 11–13] in two-dimensional (2D) Euclidean space, which belongs to the same universality class as the 2D Coulomb gas and the 2D XY model. Furthermore the SG model and its generalizations with compact variable have been thoroughly investigated in the framework of integrable field theory [14].

The fixed points and the typical scaling of ξ in their vicinity as in Eq. (1) for the 3D $O(N)$ and in Eq. (2) for the 2D SG models are located in the crossover region, i.e. between the ultraviolet (UV) and infrared (IR) regimes. It has been argued [15] that there exists an IR fixed point in the 3D $O(N)$ model at a finite momentum scale, which can be uncovered by a genuine rescaling of the couplings around the singularity of the RG evolution, i.e. where the flows stop. Recently it has been obtained that there exists a similar IR fixed point in the SG model, too [4]. We note that both IR fixed points belong to the spontaneous symmetry breaking phase of the models.

We show in this letter that this IR fixed point is accompanied by a dynamical length scale. It is defined by the degeneracy of the RG evolution, and can be identified with the reciprocal of the correlation length. This provides us a new method to determine the exponent ν beyond the crossover scaling regime in the vicinity of the IR fixed point. The IR fixed point can also account for the scaling of ξ and gives similar results that are obtained around the crossover fixed point of the model. The method is also applicable when there is no crossover fixed point, e.g. in the bi-layer SG model [16], showing a greater flexibility and a more fundamental nature of our method presented here.

II. Evolution equation.— The Wetterich RG equation for the effective action is [17]

$$\dot{\Gamma}_k = \frac{1}{2} \text{Tr} \frac{\dot{R}}{R + \Gamma_k''} \quad (3)$$

where $\dot{\cdot} = k \partial_k$, $' = \partial/\partial\varphi$ and the trace Tr denotes the integration over all momenta. Eq. (3) has been solved over the functional subspace defined by the ansatz

$$\Gamma_k = \int_x \left[\frac{Z_k}{2} (\partial_\mu \varphi)^2 + V_k \right], \quad (4)$$

with the $O(1)$ symmetric or the periodic potential V_k and the wave-function renormalization Z_k , which constant part is $z = Z_k(\varphi=0)$. The polynomial suppression has the form $R = p^2(k^2/p^2)^b$, with $b \geq 1$. Eq. (3) leads to the evolution equations for the couplings [1, 5]. The loop integral appearing in the RG equations should be performed numerically. The scale k covers the momentum interval from the UV cutoff Λ to zero. Typically we set $\Lambda = 1$. If we introduce $\bar{k} = \min(zp^2 + R)$, the RG evolution becomes singular at $k = k_f$ when

$$\bar{k}^2 - V_k''(\varphi=0)|_{k=k_f} = 0, \quad (5)$$

where $\bar{k}^2 = bk^2[z/(b-1)]^{1-1/b}$. The solution of this equation defines the scale at which the potential becomes degenerate.

II. Ising type transition.— For the 3D $O(1)$ model the potential in Eq. (4) has the form

$$V_k = \sum_{i=1}^n \frac{g_{2i}}{(2i)!} \varphi^{2i} = \sum_{i=2}^n \frac{\lambda_i}{(i)!} (\rho - \kappa)^i, \quad (6)$$

where $\rho = \varphi^2/2$ and the last equality holds up to an irrelevant additive constant. Higher precision can be achieved by the latter form of the potential, where the number $(n-1)$ couplings and the additional running parameter κ figure. One can take into account the evolution of the wave function renormalization with similar ansatz for Z_k as

$$Z_k = z + \sum_{i=1}^n \frac{z_{2i}}{(2i)!} \varphi^{2i}, \quad (7)$$

where a similar ρ dependent ansatz can be formulated. We also use the normalized couplings defined as $\bar{x} = x/\bar{k}^2$, where x can be g_{2i}, κ, z_{2i} , $i = 1, \dots, n$. Even better results can be obtained for the critical exponents without Taylor expanding the potential and the wavefunction renormalization, but at this present state it is not required. The main point here is to demonstrate the equality of the exponent ν taken from the vicinity of the WF fixed point and from the IR fixed point. In the 3D $O(N)$ model the exponent ν is identified as the negative reciprocal of the single negative eigenvalue of the matrix coming from the linearization of the evolution equations around the WF fixed point. In the lowest order of the gradient expansion, i.e. in the local potential approximation (LPA), when $Z_k = 1$, the exponent ν at the WF fixed point is $\nu \approx 0.53$ ($\nu \approx 0.64$) for $n=2$ ($n=4$), respectively [6]. We always investigated the scheme-dependence of the results by choosing different values of b . The LPA flows run into singularity at a finite scale k_f due to the strong truncation of the flow equations. The reduced temperature is identified as $t \sim (\kappa_\Lambda - \kappa_c)$ [7], i.e. the deviation of the UV initial coupling κ to its critical value κ_c , which is taken on the separatrix between the WF and the Gaussian fixed points. One can express the coupling κ by the couplings g_{2n} , e.g. for $n=2$ one has $\kappa \sim g_2/g_4$. The results of the flow show that the parameter κ blows up, i.e. $\kappa \rightarrow \infty$ at the same value of the scale \bar{k}_f [15, 18], which characterizes the degenerate potential with its minimum in the infinity, and the building up of a global condensate [19]. We identify the inverse of \bar{k}_f as the correlation length ξ . This enables us to read off the exponent ν around the IR fixed point. The LPA results can be seen in the inset of Fig. 1 for $n = 2$ and $n = 4$ couplings, which shows the coincidence of the exponents calculated from the data around the WF and the IR points, and demonstrates the possibility for finding this quantity in the IR.

By taking into account more and more couplings and including the evolution of Z_k one can push the singularity into $k=0$ giving a degenerate effective potential in the IR. Fig. 1 shows the phase diagram of the D3 $O(1)$ model projected onto the normalized couplings \bar{g}_2 and \bar{g}_4 , with the parameter choice $b = 2$. Similar phase structure can be obtained for any value of b . We numerically determined the exponent ν for $n = 8$ couplings from data

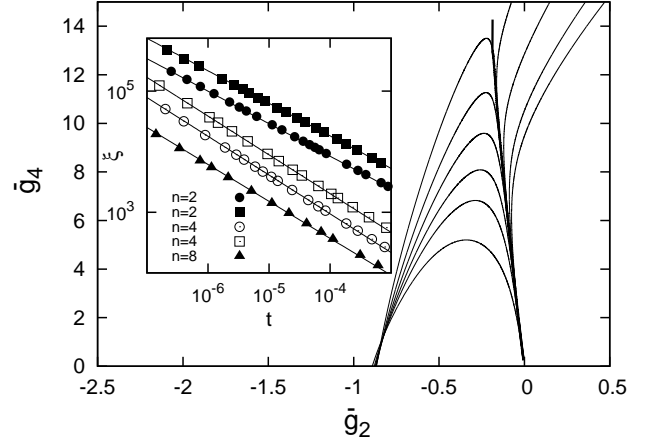


FIG. 1: The phase structure of the D3 $O(1)$ model is presented, where the evolution of Z_k is taken into account. The upper end of the thick curve denotes the WF fixed point. The trajectories tending to the right (left) correspond to the symmetric (symmetry broken) phase, respectively. The scaling of the correlation length as the function of the reduced temperature is plotted in the inset. The points are obtained from the scaling around the IR fixed point, while the solid lines represent the scaling around the WF fixed point, i.e. $\nu = 0.53$ (LPA), $\nu = 0.64$ (LPA), and $\nu = 0.62$ (including the flow of Z_k) for $n = 2, n = 4$, and $n = 8$, respectively. The curves are shifted for better visibility. The circle and square correspond to $b = 2, 5$, respectively.

around the WF and the IR points, which, similarly to the LPA results, shows high coincidence, as is demonstrated in the inset of Fig. 1. The numerical results also show that the wavefunction renormalization constant z blows up in the vicinity of the degeneracy as the function of \bar{k} , as is demonstrated in Fig. 2. The blowup of the flow of

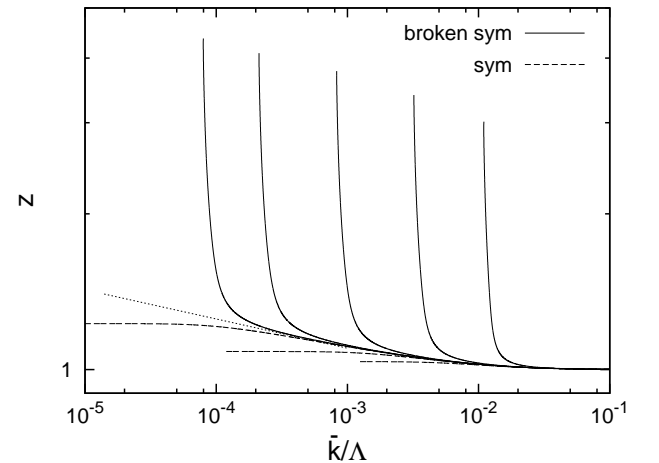


FIG. 2: The scaling of the z for $n = 8$ and $b = 2$ for several UV values of g_4 . The dotted line corresponds to the flow along the separatrix. In the deep IR regime the flows blow up in the broken symmetric phase, while they run into constant values in the symmetric one.

z occurs at smaller and smaller values of \bar{k}_f as the UV initial value κ_Λ approaches its critical value. Thus the parameters z and κ show up quite similar singular behaviour. The blowup of z results in the stop of the flows of all the couplings at \bar{k}_f in the broken symmetric phase, while in the symmetric phase z goes to a constant value, giving LPA evolution in the IR. The flow of z denoted by dotted line in Fig. 2 correspond to the flow along the separatrix in Fig. 1. Its slope gives the known, tiny anomalous dimension $\eta = -d \log z / d \log k \approx 0.05$ belonging to the WF fixed point. The flows in the vicinity of the WF fixed point pick up the effects coming from the fluctuations affected by the WF fixed point and bring them to the IR fixed point. There the anomalous dimension is extremely high (and scheme dependent), but the exponent ν must not change, since it characterizes the condensate of the broken phase. Thus, well beyond the WF fixed point scaling regime, in the vicinity of the IR fixed point in the deep IR one can recover all the information on the system.

III. Periodic model.— The 2D SG model is defined via the potential

$$V_k = u_k \cos \varphi. \quad (8)$$

The higher harmonics of the SG model are neglected. They correspond to vortices with higher vorticity of the equivalent gas of topological excitations. It is known that only the fundamental mode plays a significant role in the determination of the thermodynamical properties of the model, while effects corresponding to higher vorticity are negligible. We also note that the fundamental mode can recover the phase structure of the model including the KT transition point [11] and the IR behaviour [4]. Furthermore the wavefunction renormalization constant z can account for the KT transition [4, 10], therefore we omit further terms in Z_k . In the LPA approximation, the RG treatment of the SG model in the IR shows two phases separated by the Coleman point and the occurring the dynamical momentum scale where the evolution of the normalized coupling $\bar{u} = u/\bar{k}^2$ becomes marginal [12]. Identifying this scale as the reciprocal of the correlation length we obtain $\nu = 1$. The Coleman point becomes the KT transition point [4, 11] if Z_k evolves. Furthermore an additional IR fixed point turns up that can be transformed to the unique point at $\bar{u} = 1$ and $1/z \rightarrow 0$ when the normalized coupling \bar{u} is made of use [4]. Then any choice of b gives qualitatively similar phase diagrams. We plotted the case $b = 5$ in Fig. 3. The normalized coupling \bar{u} tends to 1 for every value of b . It shows that the degenerate potential occurs in the IR limit of the broken symmetric phase independently of the RG scheme. This reflects the serious limitation of the LPA results. In the symmetric phase the evolution of z is negligible giving the same evolution as was obtained in LPA with the line of fixed points.

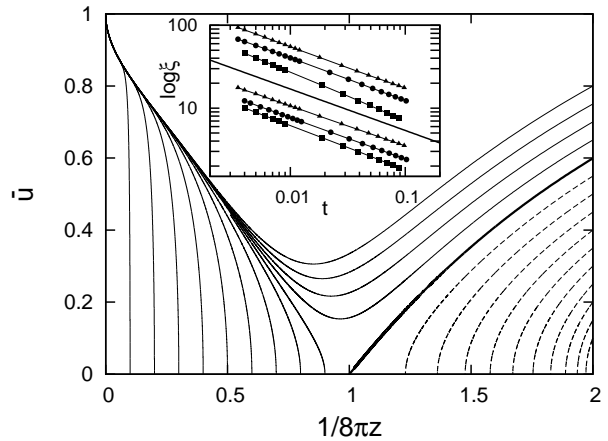


FIG. 3: Phase diagram of the SG model, with $b = 5$. The dashed (solid) lines represent the trajectories belonging to the (broken) symmetric phase, respectively. The wide line denotes the separatrix between the phases. The inset shows the scaling of the correlation length as the function of the reduced temperature t . The curves are shifted for better visibility. The lower (upper) set of lines corresponds to the IR (KT) fixed point. The triangle, circle and square correspond to $b = 2, 5, 10$, respectively. In the middle a straight line with the slope $-1/2$ is drawn to guide the eye.

One can easily show that the critical exponent η_v characterizing the vortex-vortex correlation function [20] is $\eta_v = 1/4$ independently of the parameter b [4]. However the anomalous dimension being characteristic for the divergence of the correlation function of the field variables gives $\eta = 0$ in the vicinity of the KT point. In the deep IR scaling region the situation changes significantly, there new scaling laws appear. Fig. 4 shows that around the KT point (at about $k/\lambda \sim 10^{-4}$) z is practically constant, giving $\eta = 0$, while in the IR region z diverges with the exponent $\eta = 2b/(b-1)$, giving scheme-dependent η values in the IR. It implies that one cannot avoid the evolution of z by including a phase factor $z = k^\eta$ in the LPA evolution equations. The scale \bar{k} makes the evolution of u and z qualitatively scheme-independent. The value of the wave-function renormalization z blows up at a certain minimal value of \bar{k} in the broken symmetric phase, similarly to the 3D $O(1)$ model. Likewise, the IR fixed point of the broken symmetric phase is reached at $k = 0$, but the IR scaling occurs already at some finite dynamical scale \bar{k}_f .

The initial UV value of z_Λ can be identified with the square of the temperature, thus its distance from the separatrix (for fixed u_Λ) gives the reduced temperature t . As $t \rightarrow 0$ the correlation length increases as in the case of the 3D $O(1)$ model. Our numerical results are shown in the inset of Fig. 3. There are two types of correlation lengths, one is defined as usual, namely at around the KT turning point of the coupling \bar{u} . Another one is identified

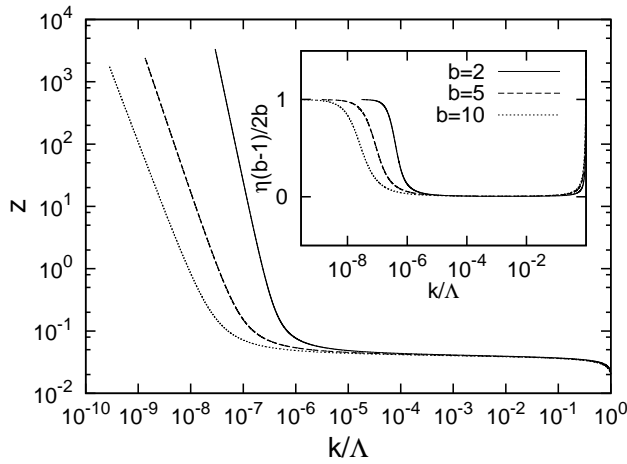


FIG. 4: The evolution of z for different values of b . The inset demonstrates that $\eta = 2b/(b-1)$ in the deep IR region.

as $\xi = 1/\bar{k}_f$ in the neighbourhood of the IR fixed point. It can be seen from the inset of Fig. 3 that the scaling of ξ shows an infinite order phase transition, for all schemes with the exponent $\nu \approx 0.5$, as opposed to the LPA result $\nu = 1$.

Such a definition ξ in the IR is also powerful when we have no crossover scaling. In the case of the bi-layer SG (LSG) model [16] one can easily show that the model has no fixed point because of the evolution of the inter-layer coupling. However, the detailed RG investigations performed by us have shown that there is an attractive IR fixed point of the LSG model with the appearing dynamical scale \bar{k}_f as in the SG model, and gives the same KT type phase transition with exponent $\nu \approx 0.57$ [21], proving the infinite nature of the phase transition there.

The scaling laws at the IR fixed point seem to coincide with those at the crossover. The IR fixed point is the signal of the spontaneous symmetry breaking in both the polynomial and the periodic models, while the KT and WF points are crossover fixed points, although they are closely related to the IR one. The dependence of the correlation length ξ on the reduced temperature has already been picked up by the trajectory in the crossover regime and is carried by it to the IR fixed point. In that sense there is an interplay between the IR and crossover points, but the proper IR physics (including both the degeneracy and the scaling of the correlation length) can only be deduced from the information on the IR fixed points. This finding clearly demonstrates the fundamental principle of the RG method, namely the IR physics should recover all the information on the system.

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